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LETTER TO THE EDITOR

Analytic treatment of the polariton problem for a smooth interface

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Abstract

We study the polariton problem for smooth boundaries, i.e. whether or not there exist some localized solutions of Maxwell's equations for different types of smooth spatial variation of complex dielectric- and/or magnetic-permittivity tensors. For a particular dielectric permittivity profile varying according to the hyperbolic tangent law, the singular term is mathematically strongly taken into account in Maxwell's equations, not in the boundary conditions. The problem is then reduced to the canonical form of Heun's equation possessing four regular singular points. The solution to Heun's equation as a power series is constructed, and an approximate solution involving a combination of two incomplete beta-functions is derived. Further, the exact eigenvalue solution to the polariton problem as a series in terms of incomplete beta-functions or, equivalently, Gauss hypergeometric functions is constructed. It is shown that the dispersion relation for the polariton wavenumber does not depend on the interface transition layer width, i.e. it is always exactly the same as the one derived in the limit of abrupt interface. We conjecture that the polariton wavenumber eigenvalue depends on either zeros of the dielectric permittivity variance profile or the poles of the logarithmic derivative of the latter.

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1. Introduction

Surface polaritons are defined as localized electromagnetic waves propagating along the interfaces of various media. A classical example is the plasmon polariton occurring at a metal–dielectric interface [1]. Exciton- [2], magnon- [3], and phonon-polaritons [4] are other important cases. From the point of view of condensed matter physics, surface waves are stipulated by the dynamics of various quasi-particles: plasmons, excitons, magnons, phonons, etc. However, from the point of view of continuous medium electrodynamics, when the light wavelength and the wave-decay distances are much larger than the interatomic distances, such localized waves can be treated phenomenologically via the macroscopic Maxwell's equations in terms of appropriate tensors of dielectric constants of the media.

Accordingly, the 'polariton problem' should be defined in the broad sense of the word: whether or not there exist some localized solutions of Maxwell's equations for different types of *smooth* spatial variation of complex dielectric- and/or magnetic-permittivity tensors. Hence, the term 'interface' is to be used merely to indicate 2D domains of space, i.e. surfaces in whose neighbourhood the above-mentioned dielectric tensors substantially vary.

For time-periodic processes occurring in linear media free from external charges the macroscopic Maxwell's equations are reduced to a system of two coupled equations symmetric with respect to the transposition of pairs $(\varepsilon, \boldsymbol{E})$ and $(\mu, \boldsymbol{H})$. The important points then are that if only one of the permittivities is spatially varied then (i) this system is split up, i.e. one of the fields, $\boldsymbol{E}$ or $\boldsymbol{H}$, can be determined independently, and (ii) the mentioned symmetry with respect to the transposition $(\varepsilon, \boldsymbol{E}) \Longleftrightarrow (\mu, \boldsymbol{H})$ is violated. These obvious observations lead to a fundamental difference in behaviour of waves with different polarizations. Mathematically, this is expressed in the fact that under such conditions the equations for TE- and TM-modes are essentially different. For instance, if the medium is isotropic, dielectric permittivity varies along only one spatial coordinate, say z', and the magnetic permittivity being everywhere constant, $\tilde{\varepsilon} = \tilde{\varepsilon}\left(z'\right)$, $\mu = \text{const}$ ($\tilde{\varepsilon}$ is the complex dielectric permittivity), then the equation for the TM-electromagnetic wave will include an additional term proportional to the logarithmic derivative of the dielectric permittivity. Despite external simplicity, this term induces an additional singularity into the problem in the regions where $\tilde{\varepsilon}(z')$ passes through zero. It is this singularity that enables the existence of waves localized in the neighbourhood of a single interface, i.e. the polariton waves!

Because of the singular nature of this term, the abrupt transition approximation (i.e. when the transient layer is assumed to be vanishingly thin) is used as a conventional approach in the polariton theory (and, generally, in the TM- and mixed-waveguide-mode theory): this singularity is actually ignored in Maxwell's equations but is taken into account indirectly via the boundary conditions [1–4]. Incidentally, when this term is discarded the basic equation is reduced to the form of the one-dimensional stationary Schrödinger equation, and there exists, of course, a well developed theory of special functions based on the hypergeometric type of equations which possess no more than three singular points. However, when retaining the term $-\partial(\ln \varepsilon)/\partial z'$, either an increase in the number of singular points or an increase of their singularity rank are expected. (Of course, this is not necessarily the case for all the possible dielectric permittivity profiles. For example, all the three planar profiles for which analytic solutions to the polariton problem have so far been found [5] present only the simplest situation, when neither additional singular points are introduced nor the singularity ranks of existing singular points are changed. Moreover, most of the profiles treated so far using approximate and/or numerical methods are also of this simple, or 'degenerate' in some sense, type—see, e.g. [5] and references therein.) This point essentially complicates the situation since, until recently, the theory of this type of equations has been developed very poorly. The only theory which, due to intensive efforts during the last decade, has presently attained a level satisfying practical requirements is that of the five equations of Heun's class (see [6] and references therein).

In the present letter, for the first time—to the best of our knowledge—we therefore treat the polariton problem for a smooth interface via Heun's equation. For a particular dielectric permittivity profile varying according to the hyperbolic tangent law, we take into account (mathematically strongly) the singular term immediately in Maxwell's equations, not in the boundary conditions. The problem is subsequently reduced to the canonical form of Heun's equation having four regular singular points. We construct the solution to Heun's equation in a power series form, and derive an approximate solution involving a combination of two incomplete beta-functions. Further, we construct the exact eigenvalue solution to the polariton problem as a series in terms of incomplete beta-functions or, equivalently, Gauss hypergeometric functions. Consequently, the dispersion relation for the polariton wavenumber is shown to be *absent* a dependence on the transition layer width, i.e. it is always exactly the same as the one derived in the limit of an abrupt interface.

2. Maxwell's equations and Heun's equation

Consider a linear medium free from external charges, so that $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$, and $\boldsymbol{j} = \sigma \boldsymbol{E}$. For time-periodic processes $\boldsymbol{E}, D, H, B \sim \exp(\mathrm{i}\omega t)$, and the macroscopic Maxwell's equations can be written in a manner symmetric with respect to the transposition of pairs $(\varepsilon, \boldsymbol{E})$ and $(\mu, \boldsymbol{H})$ form (see, e.g. [5]):

$$\triangle \boldsymbol{E} + \operatorname{grad}\left(\frac{\nabla\tilde{\varepsilon}}{\tilde{\varepsilon}}\boldsymbol{E}\right) + \frac{\omega^2}{c^2}\mu\tilde{\varepsilon}\boldsymbol{E} = -\frac{\nabla\mu}{\mu} \times \operatorname{rot} \boldsymbol{E} \tag{1}$$

$$\triangle \boldsymbol{H} + \operatorname{grad}\left(\frac{\nabla\mu}{\mu}\boldsymbol{H}\right) + \frac{\omega^2}{c^2}\mu\tilde{\varepsilon}\boldsymbol{H} = -\frac{\nabla\tilde{\varepsilon}}{\tilde{\varepsilon}} \times \operatorname{rot} \boldsymbol{H} \tag{2}$$

where the complex dielectric permittivity $\tilde{\varepsilon}$ is defined as $\tilde{\varepsilon} = \varepsilon - \mathrm{i}4\pi/\omega$. Let the dielectric permittivity vary along only the spatial coordinate z', $\tilde{\varepsilon} = \tilde{\varepsilon}\left(z'\right)$, and the magnetic permittivity be constant, $\mu = \text{const}$. The equations for TE- and TM-electromagnetic waves are thus:

TE-mode, $\boldsymbol{E} = (0, E_y, 0)$:

$$\triangle E_y + \frac{\omega^2}{c^2}\mu\tilde{\varepsilon}E_y = 0 \tag{3}$$

TM-mode, $\boldsymbol{H} = (0, H_y, 0)$:

$$\triangle H_y - \frac{1}{\tilde{\varepsilon}}\frac{\partial\tilde{\varepsilon}}{\partial z'}\frac{\partial H_y}{\partial z'} + \frac{\omega^2}{c^2}\mu\tilde{\varepsilon}H_y = 0. \tag{4}$$

Considering the solutions of equation (4) for the TM-mode presenting waves propagating along the interface in the x-coordinate direction, $H_y = H(z')\exp(\mathrm{i}k_x x)$, the equation for $H(z')$ then takes the form

$$H_{z'z'} - \frac{U_{z'}}{U}H_{z'} + (\lambda - U)H = 0 \tag{5}$$

(hereafter the alphabetical subscript denotes differentiation with respect to the corresponding variable), where, in order to unify the notations, we denote

$$\lambda = -k_x^2 \qquad U = -\frac{\omega^2}{c^2}\mu\tilde{\varepsilon}(z'). \tag{6}$$

Following the method of [7], let us transform equation (5), retaining its linearity, by addressing both dependent and independent variables:

$$H = \varphi(z)\, u(z) \qquad z' = \int \frac{\mathrm{d}z}{\rho(z)}. \tag{7}$$

This results in

$$u_{zz} + \left(2\frac{\varphi_z}{\varphi} + \frac{\rho_z}{\rho} - \frac{U_z}{U}\right)u_z + \left[\frac{\varphi_{zz}}{\varphi} + \frac{\varphi_z}{\varphi}\left(\frac{\rho_z}{\rho} - \frac{U_z}{U}\right) + \frac{\lambda - U}{\rho^2}\right]u = 0\,. \tag{8}$$

Now, we require the obtained equation to coincide with Heun's equation in its canonical form [6]:

$$u_{zz}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a}\right)u_z+\frac{\alpha\beta z-q}{z(z-1)(z-a)}u=0 \qquad (9)$$

where the parameters satisfy the Fuchsian condition, $1+\alpha+\beta=\gamma+\delta+\varepsilon$. To meet this requirement, evidently, the following should be fulfilled:

$$2\frac{\varphi_z}{\varphi}+\frac{\rho_z}{\rho}-\frac{U_z}{U}=\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a} \qquad (10)$$

$$\left(\frac{\varphi_z}{\varphi}\right)_z+\frac{\varphi_z}{\varphi}\left(\frac{\varphi_z}{\varphi}+\frac{\rho_z}{\rho}-\frac{U_z}{U}\right)+\frac{\lambda-U}{\rho^2}=\frac{\alpha\beta z-q}{z(z-1)(z-a)}. \qquad (11)$$

The analysis of the structure of these equations suggests a search for quantities ρ,φ and U in the form:

$$\varphi=z^{\alpha_1}(1-z)^{-\alpha_2}(z-a)^{\alpha_3} \qquad (12)$$

$$\rho=\frac{1}{\tau}z^{n_1}(1-z)^{n_2}(z-a)^{n_3} \qquad (13)$$

$$U=U_0z^{k_1}(1-z)^{k_2}(z-a)^{k_3} \qquad (14)$$

with τ and U_0 being arbitrary constants. The parameters γ, δ, ε are then uniquely determined from equation (10):

$$\gamma=2\alpha_1+n_1-k_1 \qquad \delta=-2\alpha_2+n_2-k_2, \qquad \varepsilon=2\alpha_3+n_3-k_3. \qquad (15)$$

Further, the existence of the independent external parameter λ (spectral parameter) in equation (11) requires the additional constraints on ρ and U:

$$\frac{\lambda\tau^2}{\rho^2}\sim\frac{p_1}{z^2}+\frac{p_2}{(1-z)^2}+\frac{p_3}{(z-a)^2}+\frac{q_1}{z(1-z)}+\frac{q_2}{z(z-a)}+\frac{q_3}{(1-z)(z-a)} \qquad (16)$$

$$\begin{aligned}U=z^{2n_1}(1-z)^{2n_2}(z-a)^{2n_3}&\left[\frac{U_1}{z^2}+\frac{U_2}{(1-z)^2}+\frac{U_3}{(z-a)^2}+\frac{V_1}{z(1-z)}\right.\\&\left.+\frac{V_2}{z(z-a)}+\frac{V_3}{(1-z)(z-a)}\right].\end{aligned} \qquad (17)$$

The parameters $\alpha_{1,2,3}$, q, α, β are then determined respectively as

$$\alpha_1(\alpha_1+n_1-k_1-1)+(\lambda p_1-U_1)\tau^2=0$$
$$\alpha_2(\alpha_2-n_2+k_2+1)+(\lambda p_2-U_2)\tau^2=0$$
$$\alpha_3(\alpha_3+n_3-k_3-1)+(\lambda p_3-U_3)\tau^2=0$$

$$\begin{aligned}q=-a[&\alpha_1(\alpha_2-n_2+k_2)+\alpha_2(\alpha_1+n_1-k_1)+(\lambda q_1-V_1)\tau^2]\\&+[\alpha_1(\alpha_3+n_3-k_3)+\alpha_3(\alpha_1+n_1-k_1)+(\lambda q_2-V_2)\tau^2]\end{aligned} \qquad (18)$$

$$\begin{aligned}-\alpha\beta=&+[\alpha_1(\alpha_2-n_2+k_2)+\alpha_2(\alpha_1+n_1-k_1)+(\lambda q_1-V_1)\tau^2]\\&-[\alpha_1(\alpha_3+n_3-k_3)+\alpha_3(\alpha_1+n_1-k_1)+(\lambda q_2-V_2)\tau^2]\\&+[\alpha_2(\alpha_3+n_3-k_3)+\alpha_3(\alpha_2-n_2+k_2)+(\lambda q_3-V_3)\tau^2].\end{aligned}$$

It is easy to see that the additional constraints (16), (17) and equations (12), (13), (14) are consistent only at certain values of $n_{1,2,3}$ and $k_{1,2,3}$. Among the allowed sets of these parameters, only a few generate physically interesting dielectric permittivity profiles. The first such set is $(n_1, n_2, n_3) = (1, 1, 0)$, $(k_1, k_2, k_3) = (0, 0, 1)$. As can readily be shown, we then obtain

$$U = U_1 + (U_2 - U_1)z \tag{19}$$

$$z = \frac{1}{1 + \exp(-\frac{z' - z'_0}{\tau})}. \tag{20}$$

Transformation (20) maps the real axis z' onto the segment $z \in [0, 1]$. In the meantime, the dielectric permittivity profile undergoes a jump according to the hyperbolic tangent law:

$$\tilde{\varepsilon} = -\frac{c^2}{\omega^2\mu}U = -\frac{c^2}{\omega^2\mu}\left(\frac{U_1 + U_2}{2} + \frac{U_1 - U_2}{2}\tanh\frac{z' - z'_0}{2\tau}\right). \tag{21}$$

As we see, the width of the transition layer is determined by the parameter τ. Now, the parameters of the problem are explicitly determined by the spectral parameter as follows:

$$\alpha_{1,2}^2 = (-\lambda + U_{1,2})\tau^2 \qquad \alpha_3(\alpha_3 - 2) = 0 \qquad a = \frac{U_1}{U_1 - U_2} \tag{22}$$

$$\gamma = 1 + 2\alpha_1 \qquad \delta = 1 - 2\alpha_2 \qquad \varepsilon = -1 \tag{23}$$

$$\alpha = \beta = \alpha_1 - \alpha_2 \qquad q = a[(\alpha_1 - \alpha_2)^2 + (\alpha_1 - \alpha_2)] - \alpha_1. \tag{24}$$

A solution of Heun's equation (9) can be constructed as a power series:

$$u = \sum_{n=0}^{\infty} a_n z^n. \tag{25}$$

For the series coefficients the following three-term recurrence relation holds:

$$\begin{aligned} &a_n a n[n - 1 + \gamma] - a_{n-1}[(1 + a)(n - 1)(n - 2) + (\gamma(1 + a) + a\delta + \varepsilon)(n - 1) + q] \\ &\quad + a_{n-2}[(n - 2)(n - 3) + (\gamma + \delta + \varepsilon)(n - 2) + \alpha\beta] = 0 \qquad n \geqslant 0 \\ &\quad a_{-2} = a_{-1} = 0 \qquad a_0 = 1 \qquad a_1 = \frac{q}{a\gamma} \end{aligned} \tag{26}$$

and the analysis of which shows that for $\mid a \mid < 1$, series (25) converges, generally speaking, only for $z \in [0, \mid a \mid]$ [6]. The solution for $z \in [\mid a \mid, 1]$ can be constructed by analytical continuation using substitution $z \dashrightarrow 1 - z$. Further, the Poincare-Perron theory of augmented convergence [6] suggests a particular value for the spectral parameter λ (and thereby the polariton wavenumber k_x) at which the series (25) converges everywhere in $z \in [0, 1]$. This value is obtained from a relation given as an infinite fraction:

$$Q_1 = \frac{R_1 P_2}{Q_2 + \frac{R_2 P_3}{Q_3 + \frac{R_3 P_4}{Q_4 + \ldots}}} \tag{27}$$

where R_n, Q_n, P_n are the coefficients of the recurrent relation (26) at a_n, a_{n-1}, a_{n-2}, respectively. However, it can be shown that, unfortunately, this value of the spectral parameter does not define a bounded (at $z = 0$ and $z = 1$) solution, such that this is not the polariton-problem eigenvalue.

Nevertheless, one can find a simple approximate solution of the problem which is valid at small thicknesses of the transition layer. This solution is easy to obtain by noting that parameters q and $\alpha\beta$ quadratically depend on τ, while the others depend linearly. Then, proceeding to the normal form of equation (9), one can be convinced that the last term of Heun's equation is small at $\tau \to 0$, and, therefore, may be neglected. In this case, the solution of equation (9) is written out in closed form using incomplete beta-functions [8]:

$$u = C_1[aB_z(-2\alpha_1, 2\alpha_2) - B_z(1 - 2\alpha_1, 2\alpha_2)] + C_2 \tag{28}$$

where one must choose $\mathrm{Re}[\alpha_1] < 0$, $\mathrm{Re}[\alpha_2] > 0$. To fulfil the boundary condition $u(z = 0) = 0$, we assign $C_2 = 0$. Then the value of $u(z)$ at $z = 1$ is expressed in terms of gamma-functions:

$$u(1) = C_1 \left(a - \frac{\alpha_1}{\alpha_1 - \alpha_2} \right) \frac{\Gamma(-2\alpha_1)\Gamma(2\alpha_2)}{\Gamma(2(\alpha_2 - \alpha_1))}. \tag{29}$$

Therefore, to achieve the last boundary condition $u(z = 1) = 0$, one should put

$$(a - 1)\alpha_1 - a\alpha_2 = 0. \tag{30}$$

As is easily verified, this condition results in the well known formula for the polariton wavenumber [1] valid in the limit of an abrupt boundary:

$$k_x^2 = \frac{\omega^2 \mu}{c^2} \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_{1+} \varepsilon_2}. \tag{31}$$

(Note that if $\alpha_{1,2}$ are real then, in order to provide $\alpha_1 < 0$ and $\alpha_2 > 0$, a must satisfy $(a - 1)/a < 0$ (see (30)), i.e. $0 < a < 1$. In terms of permittivities, $a \in [0, 1]$ is equivalent to $\varepsilon_1 \varepsilon_2 < 0$ and $\varepsilon_1 + \varepsilon_2 < 0$, which are the well known conditions for the existence of the polariton [1].)

Further, we will now show that, in fact, relation (30) exactly defines the dispersion relation for the polariton independently of τ.

3. The solution of the eigenvalue problem

In order to show that relation (30) indeed defines the eigenvalue, first note that this equation is equivalent, in our case (see (24)), to the following relation between the parameters of Heun's equation:

$$q - a\alpha\beta = 0. \tag{32}$$

Indeed, this is a crucial point since for this particular case it is possible to derive an integral representation for solutions to Heun's equation (9), namely

$$u = \int z^{-\gamma} (z - 1)^{-\delta} (z - a)^{-\epsilon} w(z)\, \mathrm{d}z \tag{33}$$

where $w(z)$ is a solution to another, modified, Heun's equation:

$$w_{zz} + \left(\frac{1 - \gamma}{z} + \frac{1 - \delta}{z - 1} + \frac{2 + \varepsilon}{z - a} \right) w_z + \frac{\alpha\beta z - q}{z(z - 1)(z - a)} w = 0. \tag{34}$$

This can be simply checked by taking the derivative of (9) and subtracting (9) (multiplied by a factor $(1/z + 1/(1 - z))$) from the obtained equation.

Now, take the power series solution to the last equation:

$$w^- = \sum_{n=0}^{\infty} a_n^- z^n. \tag{35}$$

Here the coefficients a_n^- are defined by the three-term recurrence relation (26), the parameters $\gamma, \delta, \varepsilon$ there now being replaced, according to (34), respectively, by $1 - \gamma, 1 - \delta, 2 + \varepsilon$. Substituting series (35) into (33) and integrating term-by-term we finally arrive at the following final expression in terms of combinations of incomplete beta-functions (compare with (28))

$$u^-(z) = C_1^- \sum_{n=0}^{\infty} a_n^- [a B_z(-2\alpha_1 + n, 2\alpha_2) - B_z(1 - 2\alpha_1 + n, 2\alpha_2)] + C_2^-. \tag{36}$$

Using the representation of the incomplete beta-function in terms of the Gauss hypergeometric function [8], one may express this solution in a more convenient form of hypergeometric function series. In order to meet the boundary condition $H_y(z = 0) = 0$, we put $C_2^- = 0$. Thus, a solution to the initial polariton problem, valid for $z \in [0, |a|]$, is given by

$$H_y^- = z^{\alpha_1}(1-z)^{-\alpha_2}u^-(z). \tag{37}$$

The corresponding solution for $z \in [|a|, 1]$ is written in the same way, after preliminary transformation of (34) by substitution $z \to 1 - z$:

$$u^+(1-z) = C_1^+ \sum_{n=0}^{\infty} a_n^+[(1-a)B_{1-z}(2\alpha_2 + n, -2\alpha_1) - B_{1-z}(1 + 2\alpha_2 + n, -2\alpha_1)] + C_2^+ \tag{38}$$

so that a solution to the polariton problem, valid for $z \in [|a|, 1]$, is given by

$$H_y^+ = z^{\alpha_1}(1-z)^{-\alpha_2}u^+(1-z). \tag{39}$$

Finally, demanding continuity for the solution and its first derivative at point $z = a$, we derive

$$C_1^+ = C_1^- u^-(a)/u^+(1-a) \qquad C_2^+ = 0. \tag{40}$$

It is now easy to verify that $H_y^+(z = 1) = 0$. Consequently, the derived solution, (36)–(40), presents the exact eigensolution to the polariton problem for the particular dielectric permittivity profile varying according to the hyperbolic tangent law[1]. Hence, relation (30) is the exact equation for the polariton wavenumber eigenvalue. This eigenvalue, in turn, is given by (31). As may be seen, it does not depend on the interface transition layer thickness, τ. However, the analysis of the very derivation method which led to this conclusion immediately suggests that the result extends beyond the hyperbolic-tangent model: the eigenvalue does not depend on the permittivity profile at all or, more precisely, it should depend only on some general characteristics of the profile (as the above parameter a), not the exact shape. This is the main physical result of the present paper. This conclusion does sound rather *counterintuitive* since it is basic knowledge that the behaviour of the polariton is defined by the permittivity shape only [see, e.g. [1,5]]. Obviously, more investigations are needed to clarify the situation; however, at least two immediate conjectures could explain this contradiction: (i) the polariton wavenumber eigenvalue depends on the *number and position of zeros of the permittivity profile* or (ii) it depends on the *number and position of poles of the logarithmic derivative* of the profile. It is our clear intention to test these suppositions in the near future.

4. Summary

We have shown that the polariton problem for a smooth interface having a dielectric permittivity profile varying according to an hyperbolic tangent law reduces to the canonical form of Heun's

[1] Note that at large distances from the interface the magnetic field amplitude decays exponentially along the direction perpendicular to the surface. Indeed, at $z' \to -\infty$ we have $z \sim \exp\left(\frac{z'-z_0'}{\tau}\right)$ (see (20)) so that

$$H_y^- \sim z^{-\alpha_1} \sim \exp\left(\frac{\omega}{c}\sqrt{\mu\frac{-\varepsilon_1^2}{\varepsilon_{1+}\varepsilon_2}}(z'-z_0')\right).$$

Similarly, at $z' \to +\infty$, $z \sim 1 - \exp\left(-\frac{z'-z_0'}{\tau}\right)$ and

$$H_y^+ \sim (1-z)^{\alpha_2} \sim \exp\left(-\frac{\omega}{c}\sqrt{\mu\frac{-\varepsilon_2^2}{\varepsilon_{1+}\varepsilon_2}}(z'-z_0')\right).$$

equation possessing four regular singular points, and have constructed the exact eigenvalue solution to the initial polariton problem as a series in terms of incomplete beta-functions or, equivalently, Gauss hypergeometric functions. Hence, the exact dispersion relation for the polariton wavenumber was derived, a result that *does not* depend on the transition layer width. Our conjecture is that the polariton wavenumber eigenvalue depends on either the zeros of the dielectric permittivity variance profile or the poles of logarithmic derivative of the latter.

It is noteworthy to mention that several other profiles can also be examined using Heun's equation. In particular, one such profile describes a three-layer structure which is of practical importance [1, 2], and another presents a specific four-layer system. Besides a thorough test of the above conjectures, we also hope to study the polariton problem for these profiles in the nearest future.

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